Numerical solution of 2d steady-state thermoelastic problems through a new and simple meshless Local Boundary Integral Equation (LBIE) method in combination with the Boundary Element Method (BEM)

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Keywords: Thermoelasticity, BEM, meshless LBIE method.

Abstract. A new and very simple meshless Local Boundary Integral Equation (LBIE) method is effectively combined with the Boundary Element Method (BEM) to solve steady state thermoelastic problems. The two dimensional integral equation valid for the uncoupled steady state thermoelasticity is employed for the BEM, while for the meshless LBIE representation the elastostatic fundamental solution is utilized and the thermal loading is considered as body force. The interpolation of the parameters involved in the BEM is accomplished through the use of quadratic line elements, while in LBIE method randomly distributed points without any connectivity requirement cover the analyzed domain and radial basis Functions (RBFs) are employed for the meshless interpolation of displacements and temperature. Since both methods conclude to a final system of linear equations expressed in terms of nodal displacement-temperature and tractions, their combination is accomplished directly with no further transformations as it happens in other combinations of domain methods with the BEM. Representative examples are provided in order to illustrate the achieved accuracy of the proposed here hybrid meshless LBIE/BEM formulation.

Introduction.

The majority of engineering problems in solid and structures can be considered as thermoelastic problems. Thermal stresses induced in high temperature engines, fracture and fatigue processes, interfaces, geothermal systems, pressure vessels etc., are the main concern of a numerical analysis. An accurate and robust numerical method that has been used for the solution of transient is the Boundary Element Method (BEM).The main advantages it demonstrates against other well-known numerical methods such as the Finite element method (FEM) and the Finite Differences Method (FDM), is its high accuracy and the dimensionality reduction of the problem by one, which results in a boundary only discretization of the analyzed domain [1] However, for thermoelastic problems where internal thermal sources are included, volume integrals are inserted in the integral representation of the problem rendering the BEM a domain and not a boundary only discretization method. The application of the Dual Reciprocity BEM (DR-BEM) [2] or the equivalent Particular Integrals BEM (PI-BEM) [3, 4] which transform volume integrals to boundary ones, suffer from convergence problems and questions such as which is the best Radial Basis Function (RBF) for the approximation of displacements and how many interior collocation points are necessary for acceptable accuracy [5].

The Local Boundary Integral Equation (LBIE) method proposed by Zhu et al [6] is a meshless method that employs the same integral equations with the BEM and seems to circumvent most of the aforementioned problems without any sacrifice in accuracy. It is characterized as meshless method because the interpolation is accomplished through randomly distributed points covering the domain of interest and characterized by no-connectivity requirements. The combination of the advantages of both BEM and LBIE method and the fact that both utilize the same parameters, hybrid BEM/LBIE method seems to be an excellent alternative to FEM/BEM hybrid formulations.

After the pioneering work of Zhu et al [6], meshless LBIE method has received considerable attention due to its accuracy as integral equation method and its flexibility of avoiding any kind of mesh [7,8,9].

Very recently Sellountos et al [10] proposed a stable, accurate and very simple meshless LBIE method for solving elastostatic problems, which utilizes Local Radial Basis Functions (LRBF) for the interpolation of elastic fields and its extension to three dimensions is straightforward. In the present work, that methodology is applied for two dimensional steady-state thermoelastic problems.

Integral Representation and LBIEs for steady-state thermoelastic problem.

Consider two thermoelastic materials of volume and with the same material properties and in contact to each other as it is shown in Fig 1. The steady state thermoelastic partial differential equations fulfilled for each domain are written as follows:

$$\left.\begin{aligned} &\nabla^2 T^{(1)} + \frac{1}{k} Q = 0 \\ &m\nabla^2 \mathbf{u}^{(1)} + \frac{\mu}{1-2\nu} \nabla\nabla\cdot\mathbf{u}^{(1)} - \frac{2\mu(1+\nu)}{1-2n} a\nabla T^{(1)} = 0 \end{aligned}\right\} \mathbf{x} \in \mathrm{V}^{(1)} \tag{1}$$

$$\left.\begin{aligned} &\nabla^2 T^{(2)} + \frac{1}{k} Q = 0 \\ &m\nabla^2 \mathbf{u}^{(2)} + \frac{\mu}{1-2\nu} \nabla\nabla\cdot\mathbf{u}^{(2)} - \frac{2\mu(1+\nu)}{1-2\nu} a\nabla T^{(2)} = 0 \end{aligned}\right\} \mathbf{x} \in \mathrm{V}^{(2)} \tag{2}$$

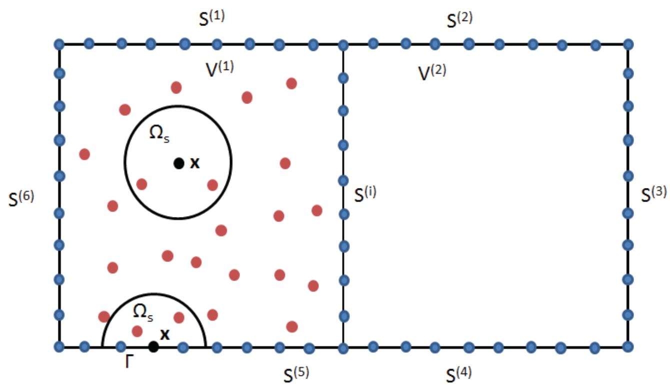

Figure 1. Coupling between two thermoelastic volumes with the same material properties.

where $T, Q, \mathbf{u}$ indicate temperature, internal thermal sources and displacement vector, respectively, while μ, ν, α, κ stand for the shear modulus, the Poisson ratio, the coefficient of linear thermal expansion and the thermal conductivity, respectively, all the same for both domains. The symbols ∇, ∇^2 represent the gradient and Laplace operators. At the global boundary $S^{(1)} \cup S^{(2)} \cup \ldots \cup S^{(6)}$ prescribed boundary conditions concerning the pairs T, q and $\mathbf{t}, \mathbf{u}$ are satisfied, while at the interface $S^{(i)}$ continuity conditions are considered. The scalar $q = -k\partial_n T$ represents thermal flux with ∂_n meaning differentiation with respect to the direction of outward normal to the boundary and vector $\mathbf{t}$ indicates field.

The solution of Eqs (2) admits an integral representation of the form:

$$cT(\mathbf{x}) - \int_S q^*(\mathbf{x},\mathbf{y})T(\mathbf{y})dS_{\mathrm{y}} = -\int_S T^*(\mathbf{x},\mathbf{y})q(\mathbf{y})dS_{\mathrm{y}} \tag{3}$$

$$cu_i(\mathbf{x}) + \int_S t_{ij}^*(\mathbf{x},\mathbf{y})u_j(\mathbf{y})dS_{\mathrm{y}} - \int_S P_i^*(\mathbf{x},\mathbf{y})T(\mathbf{y})dS_{\mathrm{y}} = \int_S u_{ij}^*(\mathbf{x},\mathbf{y})t_j(\mathbf{y})dS_{\mathrm{y}} - \int_S Q_i^*(\mathbf{x},\mathbf{y})q(\mathbf{y})dS_{\mathrm{y}} \tag{4}$$

Where all the kernels $q^*, T^*, P_i^*, Q_i^*, u_{ij}^*$ and t_{ij}^* are explicitly described in [1], while the constant c is equal to 1 for $\mathbf{x} \in V^{(2)}$ and equal to $\frac{1}{2}$ for $\mathbf{x} \in S^{(2)} \cup S^{(3)} \cup S^{(4)} \cup S^{(i)}$, except the corner points.

In the domain $V^{(1)}$, we consider a group at randomly distributed and without any connectivity requirement points as shown in Fig. 1. The points at the global boundary of $V^{(1)}$ correspond to the nodes of a BEM mesh with quadratic elements. For each internal or boundary point $\mathbf{x}$ we assume a local circular domain, centered at $\mathbf{x}$ called support domain of $\mathbf{x}$. The solution of Eqs. (1), for each support domain, is represented by the LBIEs:

$$\begin{aligned} &cT(\mathbf{x}) + \int_{\Gamma\cup\partial\Omega} \left[\partial_n\theta^*(\mathbf{x},\mathbf{y}) - \partial_n\theta^c(\mathbf{x},\mathbf{y})\right]T(\mathbf{y})dS_{\mathrm{y}} = \int_{\Gamma} \left[\theta^*(\mathbf{x},\mathbf{y}) - \theta^c(\mathbf{x},\mathbf{y})\right]q(\mathbf{y})dS_{\mathrm{y}} \\ &+ \int_{\Omega} \left[\theta^*(\mathbf{x},\mathbf{y}) - \theta^c(\mathbf{x},\mathbf{y})\right]Q(\mathbf{y})dS_{\mathrm{y}} \end{aligned} \tag{5}$$

$$\begin{aligned} &cu_i(\mathbf{x}) + \int_{\Gamma\cup\partial\Omega} \left[t_{ij}^*(\mathbf{x},\mathbf{y}) - t_{ij}^c(\mathbf{x},\mathbf{y})\right]u_j(\mathbf{y})dS_{\mathrm{y}} = \int_{\Gamma} \left[u_{ij}^*(\mathbf{x},\mathbf{y}) - u_{ij}^c(\mathbf{x},\mathbf{y})\right]t_j(\mathbf{y})dS_{\mathrm{y}} \\ &+ \frac{2\mu(1+\nu)}{1-2\nu}\alpha\int_{\Omega} \left[\partial_j u_{ij}^*(\mathbf{x},\mathbf{y}) - \partial_j u_{ij}^c(\mathbf{x},\mathbf{y})\right]T(\mathbf{y})dV_{\mathrm{y}} \end{aligned} \tag{6}$$

where θ^* is the fundamental solution of Laplace equation and θ^c, u^c, t^c are companion solutions illustrated in [6] and [8]. Finally, Γ represents the portion of the global boundary (Fig.1) when it is intersected by the support domain of the point $\mathbf{x}$.

Numerical Implementation and numerical results.

For the domain $V^{(2)}$, the standard BEM formulation described in [1] is applied. The global boundary of $V^{(2)}$ is discretized into quadratic elements and collocating integral equations (3) and (4) at all nodes, we obtain the following system of algebraic equations:

$$\left[\mathbf{H}^{(2)}\right]\cdot\begin{bmatrix}\mathbf{u}^{(2)}\\ \mathbf{T}^{(2)}\end{bmatrix} + \left[\mathbf{H}^{(21)}\right]\cdot\begin{bmatrix}\mathbf{u}^{(21)}\\ \mathbf{T}^{(21}\end{bmatrix} = \left[\mathbf{G}^{(2)}\right]\cdot\begin{bmatrix}\mathbf{t}^{(2)}\\ \mathbf{q}^{(2)}\end{bmatrix} + \left[\mathbf{G}^{(21)}\right]\cdot\begin{bmatrix}\mathbf{t}^{(21)}\\ \mathbf{q}^{(21)}\end{bmatrix} \tag{7}$$

where $(\mathbf{u}^{(2)},\mathbf{t}^{(2)},\mathbf{T}^{(2)},\mathbf{q}^{(2)})$ and $(\mathbf{u}^{(21)},\mathbf{t}^{(21)},\mathbf{T}^{(21)},\mathbf{q}^{(21)})$ are vectors containing all displacements, tractions, temperatures and fluxes defined at all nodes of the external boundary $S^{(2)} \cup S^{(3)} \cup S^{(4)}$ and the interface, respectively.

The LBIE method described in [10] is applied for the domain $V^{(1)}$. According to this method on the global boundary $S^{(1)} \cup S^{(i)} \cup S^{(5)} \cup S^{(6)}$, displacements, tractions, temperatures and fluxes are treated as independent variables. On the local domains the LBIEs (5) and (6) are applied and the local boundaries are discretized into quadratic elements. The nodal values of displacements and temperatures are interpolated via a local RBF scheme illustrated in [10]. Thus the following system of algebraic equations is obtained:

$$\left[\mathbf{H}^{(1)}\right]\cdot\begin{bmatrix}\mathbf{u}^{(1)}\\ \mathbf{T}^{(1)}\end{bmatrix}+\left[\mathbf{H}^{(12)}\right]\cdot\begin{bmatrix}\mathbf{u}^{(12)}\\ \mathbf{T}^{(12)}\end{bmatrix}=\left[\mathbf{G}^{(1)}\right]\cdot\begin{bmatrix}\mathbf{t}^{(1)}\\ \mathbf{q}^{(1)}\end{bmatrix}+\left[\mathbf{G}^{(12)}\right]\cdot\begin{bmatrix}\mathbf{t}^{(12)}\\ \mathbf{q}^{(12)}\end{bmatrix} \tag{8}$$

where $\mathbf{u}^{(1)},\mathbf{T}^{(1)}$ are vectors containing displacements and temperatures defined at all internal and boundary points except those of interface $S^{(i)}$ which are represented by the vectors $\mathbf{u}^{(12)}$ and $\mathbf{T}^{(12)}$. The vectors $(\mathbf{t}^{(1)},\mathbf{q}^{(1)})$ and $(\mathbf{t}^{(12)},\mathbf{q}^{(12)})$ are comprised of nodal values of tractions and fluxes for the nodal points lying at the global boundary $S^{(1)} \cup S^{(i)} \cup S^{(5)} \cup S^{(6)}$ and the interface, respectively.

Applying the continuity conditions at $S^{(i)}$ and the boundary conditions at the problem we conclude to a system of the form:

$$\left[\mathbf{A}\right]\cdot\mathbf{x}=\mathbf{b} \tag{9}$$

with vectors $\mathbf{x},\mathbf{b}$ containing all the unknown and known parameters of the problem, respectively. The system (9) can be solved by a standard LU decomposition procedure.

In order to demonstrate the accuracy of the just described BEM/LBIE method the following problem is solved. A hollow cylinder subjected to a thermal gradient is considered (Fig. 2). The internal and external radii of the cylinder are $a = 0.4m$ and $b = 1m$.On internal and external surfaces constant temperatures are applied. The material properties are, Poisson ratio $v = 0.25$, coefficient of thermal expansion $h = 1\text{E-}5$ /°C and thermal conductivity k= $1\text{Wm}^{-1}K^{-1}$.Because of the symmetry of the problem, only one quarter of the cylinder was modeled. Two domains, one for each method, have been considered and depicted in Fig.2. For the LBIE treatment of the problem 120 totally points have been used while for the discretization of the global boundary of the cylinder 20 quadratic elements have been considered. The results have been normalized by dividing by $P = h\theta_1$. The numerical results for radial displacements are presented in Table. 1 and are compared to those presented in [1].As it is apparent the numerical results are in excellent agreement with the corresponding analytical ones.

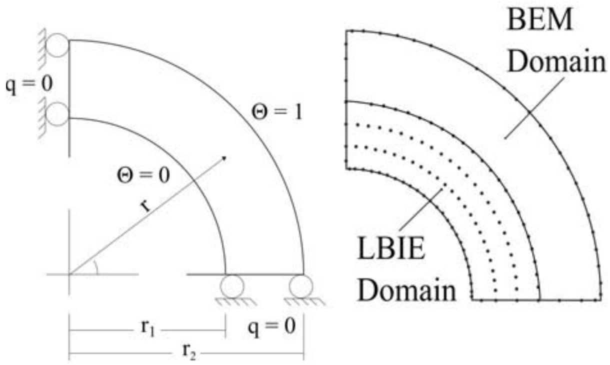

Figure. 2 Hollow cylinder subjected to a temperature gradient.

r	Numerical	Exact
0.4	0.323	0.322
0.55	0.345	0.346
0.70	0.452	0.453
0.85	0.610	0.611
1.00	0.805	0.806

Table 1.Radial displacement of the hollow cylinder.

Conclusions

A hybrid BEM LBIE method for solving 2D steady state thermoelastic problems has been proposed. The thermoelastic BEM formulation is the same with that described in Aliabadi (2002), while the LBIE methodology is based on the resent work of Sellountos et al(2012) properly modified for solving 2D thermoelastic problems.

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A detailed Boundary Element analysis of the flow field outside a growing immiscible viscous fingering within a Hele-Shaw cell

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Keywords: Viscous fingering, direct Boundary Element Method, field variable evaluation

Abstract. A BEM model for a viscous fingering problem in a Hele-Shaw cell is presented. The major point of interest in the study of this type of unstable problems is the evolution of the fluids interface where the entire nonlinear dynamic occurs. In this work besides considering the interface evolution, we will look at the corresponding flow field at each fluid region. To evaluate the flow field it is necessary to find the pressure field and its gradient inside the fluid domain. Since the reconstruction is valid only for nodes within the liquid domain, and the topology of the interface surface is typically complex, with its position constantly changing in time, it is useful to have a simple numerical scheme to determine whether or not a given location lies within the liquid or gas domain. The theory of harmonic surface potentials allow us to find such scheme, which generally is not a simple numerical task and usually very computational costly.

Introduction.

The onset and evolution of instabilities that occur in the displacement of the interface between two immiscible fluids with different viscosities is known as immiscible viscous fingering. Fingering instability appears when a fluid of higher mobility (lower viscosity) displaces a fluid of lower mobility (higher viscosity); in this case, the interface between the two fluids is unstable to perturbations of certain wavelengths, resulting in the evolution of long fingers of the less viscous fluid which penetrate into the more viscous fluid. As the fingers grow, their tips can also become unstable (tip splitting) leading to the formation of new fingers. This splitting pattern can be repeated successively as the evolution progresses, resulting in a complex interface patterns corresponding to non-linear interfacial dynamics.

The mathematical formulation of this type of interfacial problem, is described by the Darcy flow approximation (potential flow) at each side of the interface, which are matched at the sharp interface according to the continuity of normal velocity, and pressure force balanced by surface tension forces. Under these conditions the viscous instability competes with the stabilizing force of the surface tension at the interface as it is deformed by the finger (for more details see Homsy [1]).

During the evolution of viscous fingering, the interface between the two fluids experiences large deformations, and the correct determination of its shape is of paramount importance. Consequently, the use of the boundary integral equation formulation of the problem is an attractive numerical technique for its solution, and has been successfully implemented previously to study fingering problems without the effect of dissolution; see, among others, DeGregoria and Schwartz [2], Tosaka and Sugino [3], Power [4], Zhao et al. [5], Hadavinia et al. [6] and Li et al [7].

The major point of interest in the study of this type of unstable problems is the evolution of the fluids interface where the entire nonlinear dynamic occurs. For this reason little attention has been given in the literature to the corresponding flow field at each fluid region, and most of the published articles in this area only focus their analysis to the moving interface, even those based on explicit analytical solutions of the governing partial differential equations, Howison [8].

In this work a BEM model of the problem is presented, where the pressure field at the unbounded viscous fluid domain Ω is expressed as the potential field due to the injected flow plus a perturbation term $\hat{p}$. In our formulation

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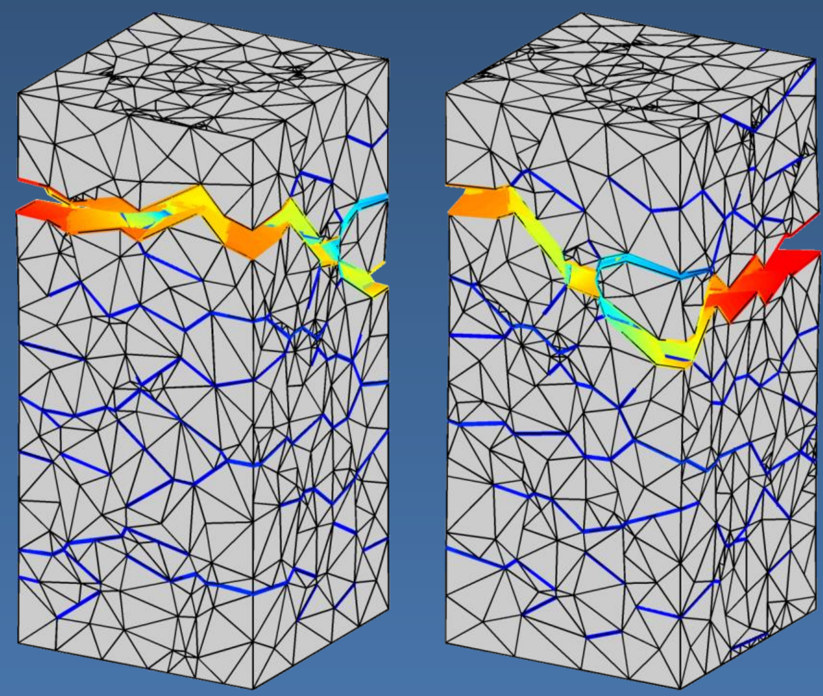

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PREFACE

The Conferences on Boundary Element and Meshless Techniques are devoted to fostering the continued involvement of the research community in identifying new problem areas, mathematical procedures, innovative applications, and novel solution techniques in both boundary element methods (BEM) and boundary integral equation methods (BIEM). Previous successful conferences devoted to Boundary Element Techniques were held in London, UK (1999), New Jersey, USA (2001), Beijing, China (2002), Granada, Spain (2003), Lisbon, Portugal (2004), Montreal, Canada (2005), Paris, France (2006), Naples, Italy (2007), Seville, Spain (2008), Athens, Greece (2009), Berlin, Germany (2010), Brasilia, Brazil (2011) and Prague, Czech Republic (2012).

The present volume is a collection of edited papers that were accepted for presentation at the Boundary Element Techniques Conference held at the LadHyX. Ecole Polytechnique, Paris, France during 16-18th July 2013. Research papers received from 18 counties formed the basis for the Technical Program. The themes considered for the technical program included solid mechanics, fluid mechanics, potential theory, composite materials, fracture mechanics, damage mechanics, contact and wear, optimization, heat transfer, dynamics and vibrations, acoustics and geomechanics.

The conference organizers would also like to express their appreciation to the International Scientific Advisory Board for their assistance in supporting and promoting the objectives of the meeting and for their assistance in the form of reviews of the submitted papers.

Editors

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